Quantum Field Theory:
A First Run Through

M.U. Sherazi

March 5, 2025

Contents

1 Preliminaries

Learning quantum field theory requires knowledge of several other fields of physics and mathematics. This section will attempt to give an overview of the information necessary for learning quantum field theory.

1.1 Classical Mechanics

Classical mechanics attempts to describe the physical behavior and motion of macroscopic entities. Newtonian physics was the first formulation of classical mechanics, and described the motion of bodies using the forces acting on them. Later formulations of classical mechanics such as the Lagrangian and Hamiltonian formulations (which describe the motion of bodies using energies) come equipped with machinery that is necessary for quantum field theory.

1.1.1 Generalized Coordinates

The physical configuration of a classical system can be described using a system of coordinates (such as Cartesian coordinates). However, for many complex problems, the use of such coordinates can become cumbersome, and this is where generalized coordinates come in handy.

Generalized coordinates are a set of coordinates which describe the configuration of a system, and are typically used to make it easier to calculate the solutions of equations of motion for the system. In a system with n degrees of freedom, an arbitrary set of generalized coordinates for the system are typically denoted by q_i, where $i = 1, \ldots, n$. Generalized velocities are denoted by $\dot{q}_i$ $(i = 1, \ldots, n)$, where the dot represents that it is the time derivative of the quantity, and the number of dots represents which order derivative (e.g. $\ddot{q}_i$ is the acceleration).

For example, you can describe the motion of a pendulum using its x and y coordinates, but the system can also be described with only one parameter if you take the angle that the pendulum makes with respect to the vertical axis, θ, as the generalized coordinate. Since the pendulum only has one degree of freedom, it makes sense that it can be described with only one parameter.

1.1.2 The Action and Euler-Lagrange Equations

In classical mechanics, the action is defined to be the following:

$$S[q_i] = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\ dt \tag{1.1}$$

where $L(q_i, \dot{q}_i, t)$ is the Lagrangian of the system, given by the difference between the kinetic energy, T, and the potential energy, V, of the system:

$$L(q_i, \dot{q}_i, t) = T(\dot{q}, t) - V(q, t) \tag{1.2}$$

In nature, the path taken by a system between two states is the one that will extremize (typically minimize) the action, S. This theorem is known as the principle of least action. Intuitively, the value of S can be thought of as a "cost" which is attributed to a given path, meaning the system will follow the path that is the most efficient/least costly.

The action is an important quantity, as calculating it for various paths allows for the true path that the system will take to be determined.

A path is mathematically represented by a time-dependent position function (i.e. $q_i(t)$). If we vary this path slightly, as to introduce a small perturbation, the new path is as follows:

$$q_{new}(t) = q_i(t) + \epsilon\eta_i(t) \tag{1.3}$$

Where ϵ is a small parameter and $\eta_i(t)$ is an arbitrary function. This also means that the generalized velocity will now be different, so that

$$\dot{q}_{new}(t) = \dot{q}_i(t) + \epsilon\dot{\eta}_i(t) \tag{1.4}$$

We will stipulate that the end points of this new path are fixed, such that $\eta_i(t_1) = \eta_i(t_2) = 0$. Now, we will calculate the action for this perturbed path:

$$S[q_{new}(t)] = \int_{t_1}^{t_2} L(q_i(t) + \epsilon\eta_i(t), \dot{q}_i(t) + \epsilon\dot{\eta}_i(t), t)\ dt \tag{1.5}$$

We will now expand the Lagrangian to first order in ϵ using the multivariate Taylor expansion, which is given by:

$$f(x, y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) + \dots \tag{1.6}$$

Because ϵ is a very small number, the higher order terms that contain ϵ^n, $n > 1$ are negligibly small, so we can ignore them. So, expanding the Lagrangian yields the following:

$$L(q_i(t) + \epsilon\eta_i(t), \dot{q}_i(t) + \epsilon\dot{\eta}_i(t), t) \approx L(q_i, \dot{q}_i, t) + \epsilon\frac{\partial L}{\partial q_i}\eta_i(t) + \epsilon\frac{\partial L}{\partial \dot{q}_i}\dot{\eta}_i(t) \tag{1.7}$$

Note that the difference terms in equation 1.6, $(x - x_0)$ and $(y - y_0)$, equal $\epsilon\eta_i(t)$ and $\epsilon\dot{\eta}_i(t)$ respectively when we apply the Taylor expansion. Now, plugging this expression back into equation 1.5 gives

$$S[q_{new}(t)] = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) + \epsilon\frac{\partial L}{\partial q_i}\eta_i(t) + \epsilon\frac{\partial L}{\partial \dot{q}_i}\dot{\eta}_i(t)\ dt \tag{1.8}$$

If we exploit the linearity of integrals across sums, we find that the first integral is simply the action, $S[q_i]$, of the original path, $q_i(t)$.

$$S[q_{new}(t)] = S[q_i] + \int_{t_1}^{t_2} \epsilon\frac{\partial L}{\partial q_i}\eta_i(t) + \epsilon\frac{\partial L}{\partial \dot{q}_i}\dot{\eta}_i(t)\ dt \tag{1.9}$$

We can see clearly now that the remaining integral represents the variation in the action due to the perturbation in the path, we will call this variation δS. As stated before, the principle of least action states that the true path a system takes is the one which makes the action extremized. A

consequence of this is that small variations in the path will not change the action. So this means that $\delta S = 0$, and we can write it as so

$$\delta S = \epsilon \int_{t_1}^{t_2} \frac{\partial L}{\partial q_i} \eta_i(t) + \frac{\partial L}{\partial \dot{q}_i} \dot{\eta}_i(t) \; dt = 0 \tag{1.10}$$

Now, if we integrate the second term by parts, we are left with the following:

$$\delta S = \epsilon \int_{t_1}^{t_2} \frac{\partial L}{\partial q_i} \eta_i(t) \; dt + \left[\frac{\partial L}{\partial \dot{q}_i} \dot{\eta}_i(t) \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}} \right] \eta_i(t) \; dt = 0 \tag{1.11}$$

Because of the boundary conditions we previously stated ($\eta_i(t_1) = \eta_i(t_2) = 0$), the second term goes to 0, and we are left with

$$\begin{aligned} \delta S &= \epsilon \int_{t_1}^{t_2} \frac{\partial L}{\partial q_i} \eta_i(t) \; dt - \int_{t_1}^{t_2} \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}_i} \right] \eta_i(t) \; dt = 0 \\ &= \epsilon \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \eta_i(t) \; dt = 0 \end{aligned} \tag{1.12}$$

Since δS is equal to 0 and we know that ϵ is a non-zero number, and $\eta_i(t)$ is an arbitrary function, the term inside of the square brackets must be 0. Thus we have:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0 \tag{1.13}$$

Equation 1.13 is known as the Euler-Lagrange equation and it is one of the important equations in physics, as it allows for the derivation of the equations of motion of a system from the Lagrangian, and thus, from the kinetic and potential energies of the system. It is equivalent to Newton's second law in classical mechanics, and lays the foundation for Lagrangian mechanics.

1.1.3 Lagrangian Mechanics

Lagrangian mechanics allows for the use of generalized coordinates, and in general is a far more flexible approach for complex problems than Newtonian mechanics.

To solve a problem in this framework, it is important to find the Lagrangian for the system, or in other words, the difference between the kinetic and potential energies of the system:

$$L = T - V \tag{1.14}$$

To see how it works in action, lets revisit the pendulum problem that was referenced in 1.1.1. As mentioned previously, the generalized coordinate that we will use will be the angle θ that the pendulum makes with the vertical axis, the bob of the pendulum also has mass m and the string is of length l. To construct the Lagrangian for this system, we will first find the kinetic energy, T:

$$\begin{aligned} T = \frac{1}{2} m v^2 &= \frac{1}{2} m (l\dot{\theta})^2 \\ &= \frac{1}{2} m l^2 \dot{\theta}^2 \end{aligned} \tag{1.15}$$

The only kind of potential energy that the bob possesses is gravitational potential energy, so V is computed as follows:

$$\begin{aligned} V &= mgh \\ &= mg(l - \cos\theta) \end{aligned} \tag{1.16}$$

So, now putting it all together, the Lagrangian is as follows:

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mg(l - \cos\theta) \tag{1.17}$$

We are now ready to solve the Euler-Lagrange equation, which for this system looks like

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0 \tag{1.18}$$

To come up with the equation of motion for the pendulum, we will solve for each of the terms in equation 1.18:

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= -mgl\sin\theta \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= ml^2\ddot{\theta} \end{aligned} \tag{1.19}$$

So, we arrive at the equation of motion for a simple pendulum:

$$\begin{aligned} ml^2\ddot{\theta} + mgl\sin\theta &= 0 \\ \ddot{\theta} + \frac{g}{l}\sin\theta &= 0 \end{aligned} \tag{1.20}$$

1.1.4 Hamiltonian Mechanics